Introduction
Port-controlled Hamiltonian systems. Modelling
Port-controlled Hamiltonian systems. Control

Port-controlled Hamiltonian Systems: modelling and control

Arnau Dòria-Cerezo
arnau.doria@upc.edu

Institut d'Organització i Control de Sistemes Industrials
UNIVERSITAT POLITÈCNICA DE CATALUNYA
Outline

1 Introduction

2 Port-controlled Hamiltonian systems. Modelling
   - Port-controlled Hamiltonian systems
   - PCHS examples

3 Port-controlled Hamiltonian systems. Control
   - Stability properties of the PCHS
   - Interconnection and Damping Assignment-PBC
   - Advances on IDA-PBC
Introduction

Port-controlled Hamiltonian systems. Modelling

Port-controlled Hamiltonian systems. Control
Affiliation

Universitat Politècnica de Catalunya
http://www.upc.edu

Teaching tasks

Department of Electrical Engineering

Research tasks

Institute of Control and Industrial Engineering (Research institute)

Advanced Control on Energy Systems (Research group)
The UPC is present in 9 cities near to Barcelona:

- Barcelona
- Canet de Mar
- Castelldefels
- Igualada
- Manresa
- Mataró
- Terrassa
- Sant Cugat del Vallès
- Vilanova i la Geltrú
Advanced Control on Energy Systems

**Keyboards:** Complex systems, non-linear systems, power systems, power electronics, control theory.

**Strategy Goals**

- **Research:** modelling and control of complex systems, and its application to problems related to the generation, conditioning, management and storage of electrical energy.

- **Technology transfer:** Diffusion of technological advances to the Industry

- **Training of specialized personnel:** Participation in
  - three **PhD programs:** Advanced Automation and Robotics, Applied Mathematics, and Applied Physics.
  - four **Master programs:** Robotics and automation, Applied mathematics, Engineering mathematics and Electronics.
Universitat Politècnica de Catalunya

- **Carles Batlle**: M.Sc. and Ph.D. degrees in Physics
- **Domingo Biel**: M.Sc. and Ph.D. degrees in Telecommunication Engineering
- **Ramon Costa-Castelló**: M.Sc. and Ph.D. degrees in Computer Science
- **Arnau Dòria-Cerezo**: M.Sc. degree in Electrical Eng. and Ph.D. in Automatic Control
- **Enric Fossas**: M.Sc. and Ph.D. degrees in Mathematics (person in charge)
- **Jaume Franch**: M.Sc. and Ph.D. degrees in Mathematics
- **Robert Griñó**: M.Sc. degree in Electrical Eng. and the Ph.D. degree in Automatic Control
- **Josep Maria Olm**: M.Sc. degree in Physics and Ph.D. in Automatic Control
- **Ester Simó**: M.Sc. and Ph.D. degrees in Mathematics

Consejo Superior de Investigaciones Científicas

- **Jordi Riera**: M.Sc. and Ph.D. degrees in Mechanical Engineering (person in charge)
- **Maria Serra**: M.Sc. degree in Physics and Ph.D. in Chemical Engineering

Multidisciplinary group
**ACES resources**

### Human resources

- **PhD**: 11
- **Lab Technicians**: 4
- **PhD students**: ~10

### ACES Labs

- Laboratory of Industrial **Power Converters**
- Laboratory of **Electrical Machines**
- Laboratory of **Fuel Cells**

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*Image showing laboratory equipment.*
Main research

- **Modelling**: Bond graph, Hamiltonian systems, linear identification, complementary systems, order reduction.

- **Analysis**: Nonlinear systems, variable structure systems, bifurcations and chaos.

- **Design**: Linear control, repetitive control, passivity-based control, flatness, sliding mode control, digital control.

- **Implementation**: DSP, FPGA, LabView, RT-Linux.

Mainly applied to electromechanical systems and fuel cells.
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3 Port-controlled Hamiltonian systems. Control
Port-controlled Hamiltonian systems... are a class of system description based on the network modelling of physical systems.

Attractive aspects...

- Linear or non-linear systems.
- Multidomain description (electrical, mechanical, thermodynamics,...).
- This formalism underscores the physics of the system: it contains the energy storage, dissipation, and interconnection structure of the system.
- The interconnection between Hamiltonian systems results in a Hamiltonian system.
- Hamiltonian systems are close to the classical Lagrangian methods, both techniques use the state dependent energy or co-energy functions to characterize the dynamics of the different elements.
Why an PCH model?

- Allows to model complex systems,
- with clear interconnection rules, and
- preserving the physical structure of the system.
- Energy-based description is an appropriate modeling framework to design passivity-based controllers.
An explicit PCHS have the form\(^1\)

\[
\begin{align*}
\dot{x} &= (J(x) - R(x))\partial_x H(x) + g(x)u \\
y &= g^T(x)\partial_x H(x)
\end{align*}
\]

where

- \(x \in \mathbb{R}^n\) is the vector state, or Hamiltonian variables.
- \(u, y \in \mathbb{R}^m\) are the port variables.
- \(H(x) : \mathbb{R}^n \rightarrow \mathbb{R}\) is the Hamiltonian function (or energy function).
- \(J(x) \in \mathbb{R}^{n \times n}\) is the interconnection matrix \((J(x) = -J(x)^T)\).
- \(R(x) \in \mathbb{R}^{n \times n}\) is the dissipation matrix \((R(x) = R^T \geq 0)\).
- \(g(x) \in \mathbb{R}^{n \times m}\) is the external port connection matrix.

\(^1\)The \(\partial_x\) operator defines the gradient of a function of \(x\), and in what follows we will take it as a column vector.
Example: DC-motor

Hamiltonian variables:
\[ x^T = [\lambda_f, \lambda_a, p], \text{ where } p = J_m \omega \]

Hamiltonian function:
\[ H = \frac{1}{2L_f} \lambda_f^2 + \frac{1}{2L_a} \lambda_a^2 + \frac{1}{2J_m} p^2 \]

Interconnection matrix:
\[ J(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -L_{af} i_f \\ 0 & L_{af} i_f & 0 \end{bmatrix} \]

PCHS form
\[ \dot{x} = (J(x) - R(x)) \partial_x H(x) + g(x)u \]

Dissipation matrix:
\[ R = \begin{bmatrix} r_f & 0 & 0 \\ 0 & r_a & 0 \\ 0 & 0 & B_r \end{bmatrix} \]

Port connection matrix:
\[ g = I_3 \]

Port input:
\[ u^T = [v_f, v_a, \tau_L] \]

Passive outputs:
\[ y^T = [i_f, i_a, \omega] \]
Example: levitation ball

Hamiltonian variables:
\( x^T = [\lambda, p, y] \), where \( p = m\dot{y} \)

Hamiltonian function:
\[
H = \frac{1}{2k} (a + y)\lambda^2 + \frac{1}{2m} p^2 - mgy
\]

PCHS form
\[
\dot{x} = (J(x) - R(x))\partial_x H(x) + g(x)u
\]

Interconnection and dissipation matrices:
\[
J - R = \begin{bmatrix}
-r & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{bmatrix}
\]

Port connection matrix:
\[
g^T = \begin{bmatrix}
1 & 0 & 0
\end{bmatrix}
\]

Port input:
\( u = v \)

Passive output:
\( y = i \)
Example: DC-DC boost converter

Hamiltionian variables:
\[ x^T = [\lambda, q] \]

Hamiltonian function:
\[ H = \frac{1}{2L} \lambda^2 + \frac{1}{2C} q^2 \]

Interconnection matrix:
\[ J = \begin{bmatrix} 0 & -s \\ s & 0 \end{bmatrix} \]

\[ s \in \{0, 1\} \]

Power converters are variable structure systems

Dissipation matrix:
\[ R = \begin{bmatrix} r & 0 \\ 0 & \frac{1}{R_l} \end{bmatrix} \]

Port connection matrix:
\[ g^T = \begin{bmatrix} 1 & 0 \end{bmatrix} \]

Port input:
\[ u = v \]

Passive output:
\[ y = i \]
Stability properties of the PCHS

Consider a PCHS

\[ \dot{x} = (J(x) - R(x)) \partial_x H(x) + g(x)u \]

with the energy function bounded from below \((H(x) > c)\), then its derivative

\[ \dot{H}(x) = (\partial H)^T \dot{x} = (\partial H)^T (J(x) - R(x)) \partial H + (\partial H)^T g(x)u \]

with \(J(x) = -J(x)^T\) and \(y = g^T(x) \partial H\) the power-balance equation is recovered

\[ \dot{H}(x) = - (\partial H)^T R(x) \partial H + y^T u \]

\[ \text{Stored power} \quad \text{Dissipated power} \quad \text{Supplied power} \]

with \(R(x) = R^T(x) \geq 0\) and considering \(u = 0\), the system is asymptotically stable

\[ \dot{H}(x) \leq 0. \]

The Hamiltonian function \(H(x)\) is a Lyapunov function.
Stability properties of the PCHS

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The Hamiltonian function \(H(x)\) is a Lyapunov function.
Consider an affine dynamical system: \( \dot{x} = f(x) + g(x)u. \)

\[
\begin{align*}
\sum_c &\quad y_c \\
+ &\quad u_c \\
\sum_I &\quad y \\
+ &\quad u \\
\sum &
\end{align*}
\]

Control by interconnection: \( \Sigma_c + \Sigma_I + \Sigma = \text{PCHS} \)

The whole system behaves as \( \dot{x} = (J_d(x) - R_d(x))\partial H_d \) where

- \( H_d(x) > c \) has a local minimum at \( x^* \),
- \( J_d = -J_d^T \) and \( R_d = R_d^T \geq 0 \).

**Matching equation**

\[
f(x) + g(x)u = (J_d(x) - R_d(x))\partial H_d
\]

**The control law**, \( u = \beta(x) \)

\[
\beta(x) = (g^T(x)g(x))^{-1}g^T(x)((J_d(x) - R_d(x))\partial H_d - f(x)
\]
Consider an affine dynamical system: \( \dot{x} = f(x) + g(x)u. \)

### Interconnection and Damping Assignment-PBC

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Consider an affine dynamical system: $\dot{x} = f(x) + g(x)u$.

The whole system behaves as $\dot{x} = (J_d(x) - R_d(x))\partial H_d$ where

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**Matching equation**

$f(x) + g(x)u = (J_d(x) - R_d(x))\partial H_d$

**The control law**, $u = \beta(x)$

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\]
Solving the matching equation...

\[ f(x) + g(x)u = (J_d(x) - R_d(x)) \partial H_d \]

Three degrees of freedom:
- \( J_d(x) \): Interconnection assignment
- \( R_d(x) \): Damping assignment
- \( H_d(x) \): Energy shaping

Proposed methods to solve the ME
- Non-Parameterized IDA
- Algebraic IDA
- Interlaced Algebraic-Parameterized IDA

There is not a best method to solve the matching equation. Each control problem requires an individual study to find out which of the above strategies provides an acceptable solution of the matching equation.
Solving the matching equation...

\[ f(x) + g(x)u = (J_d(x) - R_d(x))\partial H_d \]

Non-Parameterized IDA

- the \( J_d(x) \) and \( R_d(x) \) are fixed,
- the matching equation is pre-multiplied by a left annihilator of \( g(x) \)
  \( (g(x)^\perp g(x) = 0) \)
- and the resulting PDE in \( H_d \) is then solved.

\[ g(x)^\perp f(x) = g(x)^\perp (J_d(x) - R_d(x))\partial H_d \]

- First proposed method
- Nice and original controllers
- Difficulty on to solve some PDE
Interconnection and Damping Assignment-PBC

Solving the matching equation...

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Solving the matching equation...

\[ f(x) + g(x)u = (J_d(x) - R_d(x))\partial H_d \]

Algebraic IDA

1. the \( H_d \) function is first selected (for example a quadratic function in the error terms)
2. and then the resulting algebraic equations are solved for \( J_d \) and \( R_d \).

+ Easy solutions
+ Control law visible along the designing process
  - Feedback linearization term (Robustness)
Solving the matching equation...

\[ f(x) + g(x)u = (J_d(x) - R_d(x))\partial H_d \]

**Algebraic IDA**

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Solving the matching equation...

\[ f(x) + g(x)u = (J_d(x) - R_d(x))\partial H_d \]

**Interlaced Algebraic-Parameterized IDA**

1. the PDE is evaluated in some subspace (where the solution can be easily computed)
2. and then matrices \( J_d \), \( R_d \) are found which ensure a valid solution of the matching equation.

- Tailored solutions
- Extra degree of freedom
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Example

Advances on IDA-PBC

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**Interconnection and Damping Assignment-PBC**

---

Solving the matching equation...

\[ f(x) + g(x)u = (J_d(x) - R_d(x))\partial H_d \]

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**Interlaced Algebraic-Parameterized IDA**

1. The PDE is evaluated in some subspace (where the solution can be easily computed)

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+ Tailored solutions
- Extra degree of freedom
Example: The model

Consider the following 2-dimensional nonlinear system

\[
\begin{align*}
\dot{x}_1 &= -x_1 + \xi x_2^2 \\
\dot{x}_2 &= -x_1 x_2 + u,
\end{align*}
\]  

(1)

where \( \xi > 0 \). The control objective is to regulate \( x_2 \) to a desired value \( x_2^d \). The equilibrium of (1) corresponding to this is given by

\[
x_1^* = \xi (x_2^d)^2, \quad u^* = \xi (x_2^d)^3.
\]

This system can be cast into PCHS form

\[
\dot{x} = (J - R) \partial H + gu
\]

(2)

with

\[
J = \begin{bmatrix} 0 & x_2 \\ -x_2 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
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\[
H(x) = \frac{1}{2} x_1^2 + \frac{1}{2} \xi x_2^2.
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Example: The control design

Using the IDA-PBC technique, within the algebraic approach, we match (2) to

$$\dot{x} = (J_d - R_d) \partial H_d$$

and the Hamiltonian function is selected as

$$H_d(x) = \frac{1}{2} (x_1 - x_1^*)^2 + \frac{1}{2\gamma} (x_2 - x_2^d)^2,$$

where $\gamma > 0$ is an adjustable parameter. Interconnection and damping matrices have the form

$$J_d = \begin{bmatrix} 0 & \alpha(x) \\ -\alpha(x) & 0 \end{bmatrix}, \quad R_d = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix},$$

where $\alpha(x)$ is a function to be determined by the matching procedure and $r > 0$. From the first row of the matching equation $(J - R) \partial H + gu = (J_d - R_d) \partial H_d$ one gets

$$-x_1 + \xi x_2^2 = -(x_1 - x_1^*) + \frac{\alpha}{\gamma} (x_2 - x_2^d),$$

from which

$$\alpha(x) = \frac{\gamma}{x_2 - x_2^d} (\xi x_2^2 - x_1^*) = \ldots = \gamma \xi (x_2 + x_2^d).$$
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where \( \alpha(x) \) is a function to be determined by the matching procedure and \( r > 0 \). From the first row of the matching equation \((J - R) \partial H + gu = (J_d - R_d) \partial H_d\) one gets

\[ -x_1 + \xi x_2^2 = -(x_1 - x_1^*) + \frac{\alpha}{\gamma} (x_2 - x_2^d), \]

from which

\[ \alpha(x) = \frac{\gamma}{x_2 - x_2^d} (\xi x_2^2 - x_1^*) = \ldots = \gamma \xi (x_2 + x_2^d). \]
Example: The control design

Using the IDA-PBC technique, within the **algebraic approach**, we match (2) to

\[ \dot{x} = (J_d - R_d)\partial H_d \]

and the Hamiltonian function is selected as

\[ H_d(x) = \frac{1}{2}(x_1 - x_1^*)^2 + \frac{1}{2\gamma}(x_2 - x_2^d)^2, \]

where \( \gamma > 0 \) is an adjustable parameter. Interconnection and damping matrices have the form

\[
J_d = \begin{bmatrix}
0 & \alpha(x) \\
-\alpha(x) & 0
\end{bmatrix}, \quad R_d = \begin{bmatrix}
1 & 0 \\
0 & r
\end{bmatrix},
\]

where \( \alpha(x) \) is a function to be determined by the matching procedure and \( r > 0 \). From the first row of the matching equation \((J - R)\partial H + gu = (J_d - R_d)\partial H_d\) one gets

\[ -x_1 + \xi x_2^2 = -(x_1 - x_1^*) + \frac{\alpha}{\gamma}(x_2 - x_2^d), \]

from which

\[ \alpha(x) = \frac{\gamma}{x_2 - x_2^d}(\xi x_2^2 - x_1^*) = \ldots = \gamma \xi (x_2 + x_2^d). \]
Example: The control design

Substituting $\alpha(x)$ into the second row of the matching equation

$$-x_1x_2 + u = -\alpha(x_1 - x_1^*) - \frac{r}{\gamma}(x_2 - x_2^d),$$

yields the feedback control law

$$u = x_1x_2 - \gamma \xi (x_1 - x_1^*)(x_2 + x_2^d) - \frac{r}{\gamma}(x_2 - x_2^d). \quad (3)$$

This control law yields a closed-loop system which is Hamiltonian with $(J_d, R_d, H_d)$, and which has $(x_1^*, x_2^d)$ as a globally asymptotically stable equilibrium point.
Example: Simulations

$x_1$ and $x_2$ behaviour

\begin{figure}[h]
\centering
\begin{subfigure}{0.5\textwidth}
\centering
\includegraphics[width=\textwidth]{figure1}
\end{subfigure}\hfill
\begin{subfigure}{0.5\textwidth}
\centering
\includegraphics[width=\textwidth]{figure2}
\end{subfigure}
\end{figure}
Notice that the $\gamma$ parameter has more influence on the trajectories. This is due to the fact that $\gamma$ modifies the Hamiltonian in the $x_2$ direction and tuning this parameter makes trajectories of $x_2$ restricted (or semi-bounded).
Example: Simulations

The energy function $H_d = \frac{1}{2} (x_1 - x_1^*)^2 + \frac{1}{2\gamma} (x_2 - x_2^d)^2$
The SIDA-PBC approach suggests

\[ f(x) + g(x)u = F_d(x) \partial H_d(x) \]

where \( F_d(x) + F_d(x)^T \leq 0 \) instead of \( f(x) + g(x)u = (J_d(x) - R_d(x)) \partial H_d(x) \).

Application of SIDA-PBC also yields a closed-loop PCH system of the form

\[ \dot{x} = (J_d(x) - R_d(x)) \partial H_d(x) \]

with

\[ J_d(x) = \frac{1}{2}(F_d(x) - F_d^T(x)), \quad R_d(x) = \frac{1}{2}(F_d(x) + F_d^T(x)). \]

- The interconnection and damping matrices are fixed simultaneously.
- One restriction \( F_d(x) + F_d(x)^T \leq 0 \) instead of two \( J_d(x) = -J_d(x)^T, \quad R_d(x) + R_d(x)^T \geq 0. \)
- SIDA-PBC enlarge the set of systems that can be stabilized via IDA-PBC.
- SIDA-PBC allows to design output feedback, as opposed to state feedback, controllers.
SIDA-PBC: Example

Consider the previous dynamical system,

\[
\begin{align*}
\dot{x}_1 &= -x_1 + \xi x_2^2 \\
\dot{x}_2 &= -x_1 x_2 + u
\end{align*}
\]

where \(x_2\) is the desired output variable and \(x_1\) not-measurable.

The standard IDA-PBC the control law is...

\[
u = x_1 x_2 - \gamma \xi (x_1 - x_1^*)(x_2 + x_2^d) - \frac{x}{\gamma} (x_2 - x_2^d)
\]

which depends on \(x_1\).

Applying the SIDA-PBC approach

\[
u = x_1^* x_2 - k (x_2 - x_2^d) \quad \text{with } k \geq \frac{1}{4} \xi x_2^d
\]

we obtain a simpler and output feedback controller (only \(x_2\) measures are required).

Simulation parameters:

\(\xi = 2\), \(x_2^d = 1\) and \(k = 1\).
More advances on the IDA-PBC technique...

- **Robustness** is the main problem of the controllers obtained via IDA-PBC.
- Mainly static controllers are obtained. Some works deals on *dynamic extensions* to improve the performance and robustness.
- The basic IDA-PBC is restricted to regulation problems, *tracking* remains an open issue.
- To increase the range of applications where the IDA-PBC is used.
- ...

... are coming soon!!!
Port-controlled Hamiltonian Systems: modelling and control

Arnau Dòria-Cerezo
arnau.doria@upc.edu