Optimal Control and Applications in Operations Research and Economics

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A Deep Question

Whom will you support tonight?
Manchester United or Chelsea
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Overview

1 Brachystochrone problem
   - graphical description
   - derivation of dynamical equations
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Brachystochrone problem

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- Johann’s elder brother Jakob
- Leibniz
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5. Differential geometry techniques  post 1980
Some Interesting OR and Economic Applications

- Optimal capital spending
- Optimal reservoir control
- Optimal Production subject to Piecewise Continuous Royalty Payment Obligations
- Optimal maintenance and replacement policy
- Optimal drug bust strategy
- Optimal study for examinations
- Optimal slide presentation
Historical Development

The first optimisation result ever discovered must have been the statement that *the shortest path joining two points is a straight line segment.*
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The first optimisation result ever discovered must have been the statement that *the shortest path joining two points is a straight line segment*. The isoperimetric problem is the problem of finding, amongst all simple closed plane curves of a given fixed length, one that encloses the largest possible area. The *circle* is the shape that encloses maximum area for a given length of perimeter. It was not until the eighteenth century that a systematic theory, the *Calculus of Variations*, began to emerge.
Maxima and Minima of Functions

Let $f(x)$ be a function of the scalar variable $x$ and suppose it is defined and continuously differentiable for all $x$. To find its minimum/maximum value one must first find the points at which the first derivative

$$\frac{df}{dx} = f'(x)$$

is zero.

We look for individual values $x$ which maximize (or minimize) the function $f(x)$. 
Functions of $n$ variables

$f(x) = f(x_1, x_2, \ldots, x_n)$ with $x^T = x' = (x_1 \ x_2 \ \ldots \ x_n)$ (x a column vector) and $x \in \mathbb{R}^n$. At local maxima and minima the gradient vector of first partial derivatives is zero, i.e.

$$\frac{df}{dx} = f_x = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \begin{pmatrix} f_{x_1} \\ \vdots \\ f_{x_n} \end{pmatrix} = 0$$

We look for individual vectors $x$ which maximize (or minimize) the function $f(x)$. 
This was posed in a competition by Jean Bernoulli in 1696. The problem involves a bead sliding under gravity along a smooth wire joining two fixed points $A$ and $B$ (not directly below $A$).

What shape should the wire have so that the bead, when released from rest at $A$, slides under gravity from $A$ to $B$ in minimum time $T$?
Figure: Brachystochrone Problem

Brachystochrone problem (from the Greek *brachyst* = shortest, *chronos* = time).
Without loss of generality we can take $A$ to be the origin.

Let $B$ have coordinates $x = a$, $y = b$ and let $y = y(x)$ be the function describing the arc of the wire joining $A$ to $B$.

An element of the arc length is

$$ds = \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} dx$$

Total energy $= \frac{1}{2} mv^2 - mgy$ is conserved.
Without loss of generality we can take A to be the origin.

Let B have coordinates \( x = a, y = b \) and let \( y = y(x) \) be the function describing the arc of the wire joining A to B.

An element of the arc length is

\[
ds = \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} \, dx
\]

Total energy \( \frac{1}{2} \, mv^2 - mgy \) is conserved.

At the origin the total energy is zero since the bead is at rest.

So \( v = (2gy)^{\frac{1}{2}} \).
We need to minimise the integral

\[ \min_{y(x)} T[y(x)] = \frac{1}{(2g)^{1/2}} \int_0^a \left[ \frac{1 + \left( \frac{dy}{dx} \right)^2}{y} \right]^{1/2} dx \]  

(1)

with

\[ y(0) = 0 \quad y(a) = b \]

by suitable selection of the function (curve) \( y(x) \).
Bernoulli’s Solution

Fermat’s Minimum Time Principle and Snell’s Law for optics for a light ray

\[ v_n(x, y) = \sqrt{2gy} \]
\[ v(x, y) = \sqrt{y} \]
Bernoulli’s Solution

Fermat’s Minimum Time Principle and Snell’s Law for optics for a light ray

\[ v_n(x, y) = \sqrt{2gy} \]
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Discretize the problem into \( k \) horizontal strips of width \( \delta \).

\[ v_k = \sqrt{y_k} \quad \text{for all } k. \quad k \to \infty \text{ and } \delta \to 0 \text{ in the limit.} \]
Fermat’s Minimum Time Principle and Snell’s Law for optics for a light ray

\[
v_n(x, y) = \sqrt{2gy}
\]
\[
v(x, y) = \sqrt{y}
\]

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\[
v_k = \sqrt{y_k} \quad \text{for all} \ k \to \infty \ \text{and} \ \delta \to 0 \ \text{in the limit}.
\]

Snell’s Law

\[
\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}
\]

So

\[
\frac{\sin \theta_k}{v_k} = \frac{\sin \theta_k}{\sqrt{y_k}} = \text{constant}
\]
In the limit as $\delta \to 0$ the angle $\theta$ between the tangent to the curve and the vertical axis must be proportional to $\sqrt{y}$

$$\sin \theta = \frac{dx}{\sqrt{dx^2 + dy^2}}$$

$$\frac{y}{C} = \frac{dx^2}{dx^2 + dy^2}$$

$$\frac{C}{y} = \frac{dx^2 + dy^2}{dx^2}$$

$$\frac{C}{y} = 1 + y'(x)^2$$
\[ y'(x)^2 = \frac{C - y(x)}{y(x)} \]

\[ y'(x) = \pm \sqrt{\frac{C - y(x)}{y(x)}} \]

The parameterized curves

\[ x(\phi) = x_0 + \frac{C}{2} (\phi - \sin \phi) \]
\[ y(\phi) = \frac{C}{2} (1 - \cos \phi) \]

\[ 0 \leq \phi \leq 2\pi \]

are the **CYCLOID**.
\[ \frac{dy(x)}{dx} = \sqrt{\frac{C - y(x)}{y(x)}} \] 

since \( y'(x) \geq 0 \)

or

\[ y(x)(1 + y'(x)^2) = \text{constant} \]
\[
\frac{dy(x)}{dx} = \sqrt{\frac{C - y(x)}{y(x)}} \quad \text{since} \quad y'(x) \geq 0
\]

or
\[
y(x)(1 + y'(x)^2) = \text{constant}
\]

For any constant \(\bar{y} > 0\) \(y = \bar{y}\) (*) is a solution corresponding to \(C = \bar{y}\) with \(\frac{dy}{dx} = 0\).

So one could theoretically have the cycloid solution up to \(\phi = \pi\) with constant paths thereafter.
Euler-Lagrange Equation – A Necessary Condition

Let

$$J[y(x)] = \int_{x_0}^{x_f} L(y(x), y(x)', x) \, dx$$  \hspace{1cm} (2)$$

where $L$ and $y(x)$ belong to the class $C^2(x_0, T)$ and satisfy the boundary conditions

$$y(x_0) = A \quad \text{and} \quad y(x_f) = B$$
A necessary condition for $y(x)$ to extremize $J[y(x)]$ is that $y(x)$ satisfies the second-order differential equation

$$\frac{\partial L}{\partial y} - \frac{d}{dx} \left( \frac{\partial L}{\partial y'} \right) = 0$$

called the Euler-Lagrange Equation.
Shortest Distance between Two Points

\[ J[y(x)] = \int_0^a \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

\[ \frac{\partial L}{\partial y} - \frac{d}{dx} \left( \frac{\partial L}{\partial y'} \right) = 0 \]

\[ - \frac{d}{dx} \left( \frac{\partial}{\partial y'} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \right) = 0 \]
\[ \sqrt{1 + \left( \frac{dy}{dx} \right)^2} = C \quad \text{constant} \]

\[ \frac{dy}{dx} = D \quad \text{constant} \]

i.e. a straight line
\[ dx^2 + dy^2 = y \, dt^2 \]

\[ dt = \frac{\sqrt{dx^2 + dy^2}}{\sqrt{y}} = L(y, y') \, dx \]

\[ \frac{dt}{dx} = \frac{\sqrt{1 + \left( \frac{dy}{dx} \right)^2}}{\sqrt{y}} = L(y, y') \]
Euler-Lagrange Equation

\[ 1 + y'(x)^2 + 2y(x)y''(x) = 0 \]

The solution is satisfied by the cycloid and does NOT include the spurious solution (*).

So the solution gives better results than Bernoulli’s solution.
Control Theory

For the control system

$$\dot{x}(t) = f(x(t), u(t)), \quad x \in R^n, u \in R^m$$

find the CONTROL to make the system behave in a particular manner, e.g. follow a desired path $X(t)$, minimize a performance index (optimal control).
For $n$ curves $x(t)$ satisfying $\dot{x} = f$, the $m$ controls $u(t)$ minimise

$$J[u(t)] = \int_0^T L \, dt$$

if there exist $n$ absolutely continuous costate functions $p(t)$ and a non-negative scalar constant $p_0$ such that the Hamiltonian

$$H(x, u, p, p_0, t) = p^T f - p_0 L$$

is minimised and

$$\dot{p}(t) = -\frac{\partial H}{\partial x}$$

with suitable boundary conditions at $t = 0$ and $t = T$. 

Pontryagin’s Minimum Principle 1961
Motion in the $x$-$y$ plane.

\[
\dot{x} = u \sqrt{y} = f_1 \quad \dot{y} = v \sqrt{y} = f_2
\]

The control is the 2-dimensional vector $(u, v)$ with values in the set

\[
U = \{(u, v) : u^2 + v^2 \leq 1\}
\]

Define $p(t)$ and $q(t)$ as the scalar momentum variables conjugate to $x(t)$ and $y(t)$, and the non-zero abnormal multiplier $p_0$. 
Pontryagin’s Minimum Principle for the Brachystochrone Problem

If a curve \((x(t), y(t))\) minimises

\[ J[u(t)] = \int_0^T L \, dt \]

then there exist absolutely continuous functions \(p(t)\) and \(q(t)\) and a non-negative constant \(p_0\) such that the Hamiltonian

\[
H(x, y, u, v, p, q, p_0, t) = (pu + qv)\sqrt{y} - p_0L
\]

is minimised and

\[
\dot{p}(t) = -\frac{\partial H}{\partial x}, \quad \dot{q}(t) = -\frac{\partial H}{\partial y}
\]
In this problem we wish to minimise

\[ J[u(t), v(t)] = \int_0^T 1 \, dt \]

The optimal control can be shown to be

\[ u(t) = \frac{p(t)}{M}, \quad v(t) = \frac{q(t)}{M} \]

where

\[ M = \sqrt{p^2(t) + q^2(t)} \]

and the "adjoint system" satisfies

\[ \dot{p}(t) = 0, \quad \dot{q}(t) = -\frac{p(t)u(t) + q(t)v(t)}{2\sqrt{y(t)}} = -\frac{M}{2\sqrt{y(t)}} \]

Note that \( M \neq 0 \) and this can be shown to yield nonzero \( p_0 \).

If \( p \) vanishes, then \( \dot{x} = 0 \) and we get a vertical line.
For $\dot{x} \neq 0$.

$$y'(x) = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{v}{u} = \frac{q}{p}$$

We have

$$1 + y'(x)^2 = \frac{M^2}{p^2}$$

$$y''(x) = \frac{1}{p} \frac{dq}{dx} = \frac{\dot{q}}{p\dot{x}}$$

$$\dot{x} = \frac{p\sqrt{y}}{M}$$
\[ y''(x) = -\frac{M^2}{2yp^2} \]

So

\[ 2yy'' = -\frac{M^2}{p^2} = -(1 + y'^2) \]

and

\[ 1 + y'^2 + 2yy'' = 0 \]

leading to the cycloid solution without the spurious solution (*). Also we have not postulated that the curve has the form \( y(x) \).
Optimal Capital Spending

\[ \dot{S}(t) = \alpha S(t) - r(t) \]

\( S \) capital; \( r \) spending

\[ \max J[u] = \int_0^T e^{-\beta t} \ln(1 + r(t)) \, dt \]
Solution

\[ H = e^{-\beta t} \ln(1 + r(t)) + \lambda(\alpha S - r) \]
Solution

\[ H = e^{-\beta t} \ln(1 + r(t)) + \lambda (\alpha S - r) \]

\[ \dot{\lambda} = -\frac{\partial H}{\partial S} = -\alpha \lambda \]
Solution

\[ H = e^{-\beta t} \ln(1 + r(t)) + \lambda(\alpha S - r) \]

\[ \dot{\lambda} = -\frac{\partial H}{\partial S} = -\alpha \lambda \]

\[ \lambda(t) = Ae^{-\alpha t} \]
Solution

\[ H = e^{-\beta t} \ln(1 + r(t)) + \lambda(\alpha S - r) \]

\[ \dot{\lambda} = -\frac{\partial H}{\partial S} = -\alpha \lambda \]

\[ \lambda(t) = Ae^{-\alpha t} \]

\[ \frac{\partial H}{\partial r} = \frac{ae^{-\beta t}}{1 + r} - \lambda = 0 \]

\[ \lambda(t) = -\frac{e^{-\beta t}}{1 + r} \]

\[ r = -(e^{-\beta t} - \lambda)/\beta = -\dot{S} + \alpha S \]
\[ r(t) = Ce^{(\alpha - \beta)t - 1} \]

If \( \alpha = \beta \)

\[ r(t) = \text{constant} \]
Figure: \( a = 0.05 \) \( b = 0.05 \)
Figure: $a = 0.03$ $b = 0.02$
Figure: $a = 0.02 \ b = 0.04$
Figure: \( a = 0.02 \ b = 0.06 \)
A standard feature of the theory of the firm is that a profit maximising firm facing a downward sloping demand curve reacts to an increase in marginal cost by reducing output and increasing price.
with Kim Kaivanto (2008)

A standard feature of the theory of the firm is that a profit maximising firm facing a downward sloping demand curve reacts to an increase in marginal cost by reducing output and increasing price.

In this context, it is well understood that a requirement to pay a flat-rate royalty on sales has just this effect of increasing marginal cost and thereby decreasing output while simultaneously increasing price.
However the effect of permitting the royalty to take on more general forms has remained unaddressed to date.

Here we investigate very briefly the effect of piecewise linear cumulative royalty schedules on the optimal intertemporal production policy.
\[ \dot{y}(t) = x(t) \]

\( y(t) \) cumulative production up to time \( t \)

\( x(t) \) production at time \( t \) [control variable]
\[ \dot{y}(t) = x(t) \]

\[ y(t) \quad \text{cumulative production up to time } t \]

\[ x(t) \quad \text{production at time } t \text{ [control variable]} \]

Maximise by selection of optimal \( x(t) \)

\[
J[x(t)] = \int_0^T \left\{ [a(t)x^{-\alpha} - (m_0 + \rho + c_0e^{-\lambda y})]x(t)e^{-rt} \right\} dt
\]

\[
= \int_0^T g(y, x, t) dt
\]

\[ a(t)x^{-\alpha} \quad \text{demand curve and } r \text{ is the discount rate} \]

\[ m_0 + c_0e^{-\lambda y} \quad \text{learning curve (unit cost for producing one unit of output)} \]

\[ \rho(x(t), y(t), t)x(t) \quad \text{total royalty paid at time } t \]
Royalty $\rho$ is piecewise continuous

$y(T)$ the cumulative production up to time $T$ is unknown \textit{a priori}

If $y(t) < 0.08y(T)$, $\rho = 0$
If $y(t) > 0.08y(T)$, $\rho = 1.2$
If $y(t) > 0.16y(T)$, $\rho = 1.6$
If $y(t) > 0.24y(T)$, $\rho = 2$
If $y(t) > 0.4y(T)$, $\rho = 2.4$
If $y(t) > 0.56y(T)$, $\rho = 0.24$
If $y(t) > 0.72y(T)$, $\rho = 0.12$
The optimal solution satisfies

\[ H(y, x, p, t) = x(t) + p(t)g \]

\[ \dot{p} = -\frac{\partial H}{\partial y} \]

\[ \max_{x(t)} H(y, x, p, t) \text{ by setting } \frac{\partial H}{\partial x} = 0 \]

subject to \( y(0) = y_0 \) and \( p(T) = 0 \)

This is a Two Point Boundary Value Problem [TPBVP]

but additionally \( y(T) \) in the revenue function is not known \textit{a priori}. 
Solve using the **Shooting Method**.

From an initial guessed value $p(0)$ at $t = 0$
and also a guessed value $y(T) = z(0)$ with $\dot{z} = 0$,
iterate by solving the 3-dimensional ode system
$[y(t), p(t), z(t)]^T$ until $p(T) = 0$ and $z(0) = y(T)$

Note that $z(t) = \text{constant}$.

This is solved using C++ or SciLab.
Figure: Cumulative production
Figure: Production
How long should one keep a motor car, knowing that without maintenance expenditure $u(t)$ the car will deteriorate at a rate $\delta$. Beyond a certain age the car will not be worthwhile to repair.

$$\dot{x}(t) = -\delta x(t) + u(t)g(t)$$

where $g(t)$ is the maintenance effectiveness function. One wishes to maximise

$$J[u(t)] = \int_{0}^{T} \{wx(t) - u(t)e^{-rt}\} dt + x(T)e^{-rt}$$

where $w > 0$ and the discount factor $r > 0$ and

$$0 \leq u(t) \leq M, \quad M > 0$$

$T$ and $x(T)$ are free.
$N(t)$ number of dealers at time $t$

$\omega_0$ equilibrium level of the generalised profit
Optimal Crackdown on Illicit Drug Markets [Baveja, Feichtinger, Hartl et al]

\( N(t) \) number of dealers at time \( t \)
\( \omega_0 \) equilibrium level of the generalised profit
If generalised profit > \( \omega_0 \) dealers enter
\( \pi(t) \) generalised profit per unit of sales in the market
\( \alpha N^\beta \) = number of sales per day with \( N \) dealers
\( E(t) \) enforcement effort with crackdown at time \( t \)
generalised net profit with per dealer cost of enforcement is
\( \pi \alpha N^\beta / N - \left( \frac{E}{N} \right)^\gamma \)

\[ \dot{N} = c_1 \left[ \pi \alpha N^{\beta-1} - \left( \frac{E}{N} \right)^\gamma - \omega_0 \right] \]

where \( c_1 \) is the speed of adjustment parameter
$B$ budget available for the crackdown
$T$ horizon date
$r$ discount rate
minimise $e^{-rT}N(T)$ subject to

$$\dot{N}(t) = c_1 \left[ \pi \alpha N^{\beta-1} - \left( \frac{E}{N} \right)^\gamma - \omega_0 \right]$$

with $N(0) = N_0$ and $\int_0^T E(T)dt \leq B \quad (*)$. Replace (*) by an isoperimetric constraint

$$\dot{D} = E, \quad D(0) = 0$$

by adding a variable $D(t) \leq B$, money spent on crackdown enforcement in $[0, t)$.
Two states $N(t)$ and $D(t)$, and one control $E$. 
Hamiltonian

\[ H = c_1 \left[ \pi \alpha N^{\beta - 1} - \left( \frac{E}{N} \right)^\gamma - \omega_0 \right] + \lambda_2 E \]

\[ H_E = -\lambda_1 x_1 E^{\gamma - 1} N^{-\gamma} + \lambda_2 = 0 \]

\[ \dot{\lambda}_1 = r \lambda_1 - H_N \]

\[ \dot{\lambda}_2 = r \lambda_2 - H_D = r \lambda_2 \]

\[ \lambda_1(T) = -1, \quad \lambda_2(T) = -\alpha \quad \alpha \geq 0, \quad \alpha(B - D(T)) = 0 \]
Two Scenarios

Risk-Seeking Dealers — decreasing returns to scale of enforcement

\[ \gamma < 1 \]
\[ E^*(t) \text{ increases with } t \]

Non-Risk-Seeking Dealers — constant and increasing returns to scale of enforcement

\[ \gamma \geq 0 \]
\[ E^*(t) \text{ at a maximum until the market collapses} \]
Optimal Study Problem

\[ \frac{dK}{dt} + cK = aK^p W^r \]

\( K \) knowledge level attained
work effort \( W \leq W_m \) less than \( W_m \)

\[ 0 \leq p \leq 1 \quad 0 \leq r \leq 1 \]

c the student forgets a constant portion of what he/she knows
To pass the module a student must attain a given knowledge level \( K_T \).

\[ \max_{W(t)} J[W(t)] = \int_0^T (Ke^{ct})^{1-p} dt \]
$N$ total number of slides prepared for a lecture (assumed fixed)
y(t) cumulative number of slides presented at time $t$
$0 \leq t \leq T$
x(t) remaining number of slides at time $t$
x = $N - y$; \quad x(0) = N \text{ and } y(0) = 0

$$\dot{x} = -\dot{y}$$

Let

$$\dot{y} = u, \quad u(t) \text{ presentation rate}$$
Assume $T$ is free and can be selected suitably.

(i) Let $\bar{T}$ be the official length of the lecture
- Introduce a penalty term $c(T - \bar{T})^2$
- Alternatively no penalty if $T < \bar{T}$

(ii) The chairman is very strict so $T \leq \bar{T}$
- Let $0 \leq u \leq u_m$ constant
- $x(T) \geq T$ so $y(T) \leq N$
Let $k$ (positive constant) be the average effort to prepare a slide.

$kx(T)$ wasted effort on slides not presented.

Utility function $U = ay - by^2 \quad a, b \geq 0$

$U > 0$ for $0 < y < a/b$

$dU/dt > 0$ for $0 < y < a/(2b)$

Increasing the number of slides beyond a reasonable limit creates bad will in most audiences.
To maximise the utility function but minimise wasted effort

$$\max_{u,T} -kx(T) + \int_0^T (ay - by^2) dt$$

There are essentially three controls; $u(t)$, $T$ and $N$. $T$ and $N$ should be selected at $t = 0$. 
To maximise the utility function but minimise wasted effort

$$\max_{u,T} -kx(T) + \int_{0}^{T} (ay - by^2) dt$$

There are essentially three controls; $u(t)$, $T$ and $N$. $T$ and $N$ should be selected at $t = 0$.

It is convenient to set $N(0)$ as a degenerate state variable

$N(0)$ free and $N(T)$ free and $\dot{N} = 0$
(a) Case of a rigid chairman \( T \leq \bar{T} \)

\[
\max_{u,T} \int_{0}^{T} (ay - by^2)dt - kx(T)
\]

subject to

\[
\dot{x} = -u \\
\dot{N} = 0 = N_t \\
x(0) = N(0) \\
T \in [0, \bar{T}] \\
0 \leq u \leq u_m \\
x(T) \geq 0, \quad N(T)\text{free}
\]
Proposition 1: $T^* = \bar{T}$

Proposition 2: $x^* (T^*) = 0$
The Hamiltonian is

\[ H(x, N, u, \lambda_0, \lambda, \mu) = \lambda_0(a(N - x) - b(N - x)^2) - \lambda u + \mu N_t \]

Let \((x^*(t), N^*(t), u^*(t))\) be an optimal triple.
The Hamiltonian is
\[
H(x, N, u, \lambda_0, \lambda, \mu) = \lambda_0(a(N - x) - b(N - x)^2) - \lambda u + \mu N_t
\]

Let \((x^*(t), N^*(t), u^*(t))\) be an optimal triple.
There exists a constant \(\lambda_0 \geq 0\).
Let \(\lambda_0 = 1\) say.
There exist piecewise continuously differentiable costates \(\lambda(t), \mu(t)\) and constant multipliers \(\alpha, \beta\) such that \((\lambda_0, \lambda, \mu, \alpha, \beta) \neq 0\) except at discontinuous points of \(u^*\)
\[ H(x^*, N^*, u^*, \lambda_0, \lambda, \mu) = \max_{0 \leq u \leq u_m} H(x^*, N^*, u^*, \lambda_0, \lambda, \mu) \] (3)

\[ H^* = \lambda_0(a(N^* - x^*) - b(N^* - x^*)^2) - \lambda u^* = \text{constant} \] (4)

\[ \dot{\lambda} = -H^*_x = \lambda_0(a - 2b(N^* - x^*)) \] (5)

\[ \dot{\mu} = -H^*_N = \lambda_0(a - 2b(N^* - x^*)) \] (6)

\[ - \begin{bmatrix} \lambda(0) \\ \mu(0) \end{bmatrix} = \beta \frac{\partial(x(0) - N(0))}{\partial(x, N)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \beta \] (7)

\[ \lambda(T) = -\lambda_0 k + \alpha; \quad \mu(T) = 0 \] (8)

\[ \alpha \geq 0, \; \alpha x^*(T^*) = 0 \] (9)
\[ H^* = \lambda_0 (a(N^* - x^*(T^*)) - b(N^* - x^*(T^*))^2) - \lambda(T^*)u^*(T^*) \]

\[
\begin{cases}
\leq & \{ 0 \} \\
= & \text{if} \quad \begin{cases}
T^* = 0 \\
0 < T^* < \bar{T} \\
T^* = \bar{T}
\end{cases}
\end{cases}
\]

Because of Proposition 1 \( H(T^*) \geq 0 \)

From (6), (7), (8)

\[ \int_0^T H_N dt = - \int_0^T \mu dt = \mu(0) - \mu(T) = \mu(0) = \beta \quad (10) \]
From (7) $\lambda(0) = -\mu(0) = -\beta$ so

$$\int_0^T H_N dt = - \int_0^T H_x dt = \int_0^T \dot{\lambda} dt$$

$$= \lambda(T) - \lambda(0) = \lambda(T) + \beta$$

and from (10) $\lambda(T) = 0$. Also $\alpha = k > 0$, and $x^*(T) = 0$. Let $\lambda_0 = 1$. 
To maximise $H$ wrt $u$

$$u^* = \begin{cases} u_m & \text{if } \lambda < 0 \\ \text{singular} & \text{if } \lambda = 0 \\ 0 & \text{if } \lambda > 0 \end{cases}$$

The optimal control has $u^* = u_m$ for $t \in [0, \tau]$ where

$$\tau = \frac{a}{2bu_m}.$$

The singular path occurs when $\lambda = 0$; $x_s = N - \frac{a}{2b}$ with $u_s = 0$ in the interval $t \in [\tau, T]$.

This problem can be interpreted as a non-renewable resource problem.
More realistic problems include
(b) Sloppy chairman: we disregard the constraint $0 \leq T \leq \bar{T}$
The optimal solution is similar to the above but now

$$T^* = \bar{T} + \frac{a^2}{8bc} > \bar{T}$$
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(c) **Forgetful audience**: $\dot{y} = u - \delta y$  $\delta > 0$
$\delta$ may be regarded as the number of slides still kept in memory at $t$ by an average member of the audience.
Similar result but now $u_s > 0$ (a more realistic result).
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(d) Stochastic chairman and presenter
Probabilistic parameters.
THANK YOU FOR YOUR PATIENT ATTENTION